STABILITY OF AN AXISYMMETRIC JET OF MAGNETIZABLE FLUID

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INTRODUCTION

It is known (e.g., [1]) that the stability condition for jet flow of an ideal fluid is determined in the final analysis by the nature of the pressure distribution in the fluid and its jump at the free surface of the jet. Since the pressure distribution in a magnetizable fluid and the pressure jump at its free surface are determined not only by hydrodynamic quantities, but also by the magnetic field, one must assume that the stability condition for a jet of magnetizable fluid will be significantly different from that for a similar jet of nonmagnetic fluid. Surface waves and the stability of a plane free surface of a stationary infinitely deep magnetizable fluid have been considered in a number of papers [2-5]. These papers show that a free surface in a magnetic field normal to it is unstable, determine the critical value of the magnetic field intensity, and explain the stabilizing effect of a tangential magnetic field on surface perturbations propatating along it. The mechanism of the interaction of field perturbations with an infinitely deep fluid having a plane surface which leads to instability or stabilization of surface perturbations should manifest itself in jet flows as well as in flows with a free surface, but the critical value of the field and the critical wavelength are affected not only by the parameters determining the stability of a plane surface of an infinitely deep fluid, but also by the initial curvature of the surface and the finite thickness of the jet. The classical problem of the stability of an axissymmetric vertical jet of ideal fluid was treated at the end of the last century by Rayleigh [6], who showed that such a jet is always unstable: Any perturbation with a wavelength greater than the perimeter of the jet increases with time. The instability is caused by surface-tension forces of the cylindrical surface. The force acting on a unit surface of a magnetizable fluid is determined not only by the surface tension, but also by the magnetic field. In addition, a nonuniform magnetic field leads to a nonuniform pressure distribution in a magnetizable fluid, and this has a substantial effect on the stability of the surface of the fluid. Thus, the stability of an axisymmetric jet of magnetizable fluid in a magnetic field must be substantially different from that of a similar jet of nonmagnetic fluid.

1. Statement of the Problem

Neglecting thermal and electrical conductivities and internal angular momentum, the motion of an ideal magnetizable fluid is described by the equations [7, 8]

$$\rho\,[\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v}] = -\nabla p + \rho\mathbf{g} + \mu_0 M\nabla H,\ \operatorname{div}\mathbf{v} = 0,$$
$$\operatorname{rot}\mathbf{H} = 0,\ \operatorname{div}\mathbf{B} = 0, \tag{1.1}$$
$$\mathbf{B} = \mu\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M}),\quad \mathbf{M} = [M(H)/H]\mathbf{H}$$

with the following boundary conditions at the free surface separating media 1 and 2 [9]:

$$\{p + (1/2)\mu_0(\mathbf{M}\cdot\mathbf{n})^2\} = \alpha(1/R_1 + 1/R_2),$$
$$\{\mathbf{B}\cdot\mathbf{n}\} = 0,\ \{\mathbf{H}\times\mathbf{n}\} = 0, \tag{1.2}$$

where $\{a\} = a_1 - a_2$, $\mathbf{M}$ is the magnetic moment per unit volume of the magnetizable fluid, $\mathbf{B}$ and $\mathbf{H}$ are, respectively, the magnetic induction and the magnetic field intensity in the fluid, $\mathbf{n}$ is a vector normal to the free surface, p includes the hydrostatic pressure and the pressure arising from the magnetostrictive effect (cf., e.g., [4]), α is the surface tension, and R_1 and R_2 are the principal radii of curvature.

We consider a vertical cylindrical jet of magnetizable fluid having a radius a and a permeability μ surrounded by a medium whose density and magnetization can be neglected. The jet is in an external magnetic field $\mathbf{H}_2 = \{H_{2r}(r), H_{2\theta}(r), H_{2z}\}$ which satisfies Maxwell's equations and does not disturb the axial symmetry of the jet. Here r, θ, and z are cylin-

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drical coordinates with the z axis along the axis of the jet. An example of such a field is $\mathbf{H}_2 = \{A/r,\ B/r,\ H_{2z} = \text{const}\}$*.

We study the stability of the jet described above in the linear approximation in small perturbations of the velocity $\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$, the pressure $p' = p - p_0$, the magnetic field intensity $\mathbf{h}_{1,2} = \mathbf{H}_{1,2} - \mathbf{H}^0_{1,2}$, $H'_{1,2} = H_{1,2} - H^0_{1,2}$ and the magnetization $\mathbf{m}_{1,2} = \mathbf{M}_{1,2} - \mathbf{M}^0_{1,2}$, $M'_{1,2} = M_{1,2} - M^0_{1,2}$, where a superscript 0 denotes the equilibrium value.

We transform to a reference system in which the jet is at rest. Then, introducing the velocity perturbation potential $\mathbf{v}' = -\nabla\varphi$ and the magnetic field perturbation potential $\mathbf{h}_{1,2} = -\nabla\Psi_{1,2}$, where the subscripts 1 and 2 refer, respectively, to the fields inside and outside the jet, and linearizing Eqs. (1.1) and boundary conditions (1.2), we obtain the equations for the potentials of small perturbations, assuming the magnetization is linearly proportional to the magnetic field $\mathbf{M} = \chi\mathbf{H}$,

$$\Delta\varphi = 0,\ \Delta\Psi_1 = 0,\ \Delta\Psi_2 = 0,$$

whose solutions must satisfy the conditions at the free surface

$$\begin{gathered}
\rho\frac{\partial\varphi}{\partial t} + (\mu-\mu_0)H_1\frac{\partial H_1}{\partial r}\zeta + (\mu-\mu_0)(\mathbf{H}_1\cdot\mathbf{h}_1) = \frac{(\mu-\mu_0)^2}{\mu_0}\Big(-H_{1r}\frac{\partial H_{1r}}{\partial r}\zeta - \\
- H_{1r}h_{1r} + H_{1z}H_{1r}\frac{\partial\zeta}{\partial z} + \frac{1}{a}H_{1r}H_{1\theta}\frac{\partial\zeta}{\partial\theta}\Big) + \alpha\Big[\frac{1}{a} - \frac{1}{a^2}\Big(\zeta + \frac{\partial^2\zeta}{\partial\theta^2}\Big) - \frac{\partial^2\zeta}{\partial z^2}\Big], \\
\frac{\partial\zeta}{\partial t} = -\Big(\frac{\partial\varphi}{\partial r}\Big)_{r=a},\quad \mu_0 h_{2r} - \mu_1 h_{1r} = (\mu-\mu_0)\Big(H_{2z}\frac{\partial\zeta}{\partial z} + \frac{1}{a}H_{2\theta}\frac{\partial\zeta}{\partial\theta}\Big), \\
h_{2z} - h_{1z} = \frac{\mu-\mu_0}{\mu_0}H_{2r}\frac{\partial\zeta}{\partial z},\ h_{2\theta} - h_{1\theta} = \frac{\mu-\mu_0}{\mu_0}H_{2r}\frac{\partial\zeta}{\partial\theta}\frac{1}{a}
\end{gathered} \tag{1.3}$$

and the conditions that the perturbations are finite at r = 0 and damped as $r \to \infty$. Here $\zeta = r - a$ is the small radial deviation of points of the free surface of the jet from the equilibrium cylindrical surface. The boundary conditions require that the equilibrium values of the fields satisfy $H_{2\theta} = H_{1\theta}$, $H_{2z} = H_{1z}$, $\mu_0 H_{2r} = \mu H_{1r}$ at $r = a$. Here, as in (1.3) and in all that follows, the zero superscripts denoting equilibrium values are omitted.

2. The Dispersion Equation and Its Analysis

Since the form of the solution of the Laplace equation in cylindrical coordinates is critically dependent on whether it is periodic along the axis of symmetry, we write the surface perturbations as a sum of two independent terms, $\zeta = \zeta_1 + \zeta_2 = \delta_1 \exp(is\theta - i\omega t) + \delta_2 \exp(il\theta + ik_z z - i\omega t)$, where the first term is periodic only in the angle $\theta(k_z = 0)$ and the second in both θ and z. As a result of the periodicity of the solution in θ, s, and l are integers which independently take on all values of $s > 0$ and $l \geqslant 0$ ($s = 0$ corresponds to the trivial solution $\varphi = \text{const}$). The linear formulation of the problem logically leads to the assumption that the perturbations of physical quantities are proportional to the surface perturbations, so we write them in the form $f(r, \theta, z) = f_1(r)\zeta_1 + f_2(r)\zeta_2$. The general solution of the Laplace equation for such perturbations is $f_1 = a_1 r^s + a_2 r^{-s}$, $f_2 = a_3 I_l(k_z r) + a_4 K_l \cdot (k_z r)$, where I_l and K_l are modified Bessel functions and the a_i are coefficients determined from the boundary conditions. The substitution of this solution into the boundary conditions leads to two independent dispersion equations,

$$\begin{gathered}
\rho\omega^2 = \frac{s}{a}\Big[-(\mu-\mu_0)H_1\frac{\partial H_1}{\partial r} - \frac{(\mu-\mu_0)^2}{\mu_0}H_{1r}\frac{\partial H_{1r}}{\partial r} + \\
+ \alpha\frac{s^2-1}{a^2} + \frac{s}{a}\frac{(\mu-\mu_0)^2}{\mu_0(\mu+\mu_0)}\left(-H_{1r}^2\mu + H_{1\theta}^2\mu_0\right)\Big];
\end{gathered} \tag{2.1}$$

$$\begin{gathered}
\rho\omega^2 = k_z\frac{I'_l(k_z a)}{I_l(k_z a)}\Big\{-(\mu-\mu_0)H_1\frac{\partial H_1}{\partial r} + \\
+ k_z\frac{(\mu-\mu_0)^2\left[\mu K'_l I'_l H_{1r}^2 + \mu_0 K_l I_l\left(H_{1z} + lH_{1\theta}^2/k_z a\right)^2\right]}{\mu_0\left(\mu K_l I'_l - \mu_0 K'_l I_l\right)} - \\
- \frac{(\mu-\mu_0)^2}{\mu_0}H_{1r}\frac{\partial H_{1r}}{\partial r} + \alpha\frac{k_z^2 a^2 + l^2 - 1}{a^2}\Big\},
\end{gathered} \tag{2.2}$$

*The stability of a jet of magnetic fluid in a solenoidal field $\mathbf{H}_2 = \{0,\ 0,\ H_{2z} = \text{const}\}$ was investigated in [12].

describing the behavior of the perturbations when $k_z = 0$ [Eq. (2.1)] and $k_z \neq 0$ [Eq. (2.2)]. In both equations the values of the field intensity and its gradient are taken at $r = a$.

The presence of a magnetic field gradient directed toward the center, which generally occurs in axisymmetric fields, leads to a compression of the cylindrical column of fluid which is somewhat similar to the pinch effect for electrically conducting fluids (plasmas) [10]. However, the effect of the magnetic field on the stability of this column is completely different. As a matter of fact, as already noted above, a cylindrical jet of ordinary fluid is always unstable, while it follows from Eq. (2.2) that an axisymmetric jet of magnetizable fluid is unstable if

$$G \equiv -(\mu-\mu_0) H_1 \frac{\partial H_1}{\partial r} - \frac{(\mu-\mu_0)^2}{\mu_0} H_{1r} \frac{\partial H_{1r}}{\partial r} - \frac{\alpha}{a^2} < 0 \tag{2.3}$$

(only in this case a number k_z can always be found which is small enough so that for $l = 0$ the expression in curly **brackets** is negative and $\omega^2 < 0$, which also leads to instability), that is, the presence of a sufficiently large field gradient directed toward the center stabilizes a cylindrical volume of magnetic fluid, while a self-constricted plasma pinch (pinch effect) is always unstable. In its effect on stability, the quantity on the left-hand side of inequality (2.3) plays the same role for a cylindrical column of fluid as $(\rho_1 - \rho_2)g$ does for a plane surface of separation of two fluids. Thus, the behavior of the Rayleigh–Taylor instability of a plane surface is determined by the condition $(\rho_1 - \rho_2)g < 0$, where ρ_1 is the density of the lower fluid, whereas the analogous condition (2.3) determines the instability of a cylindrical surface.

Continuing the investigation of Eq. (2.2), we note that the nonfulfillment of Eq. (2.3) is a necessary but not sufficient condition to ensure the stability of an axisymmetric jet. Since $K_l'/K_l < 0$ and $I_l'/I_l > 0$, for a sufficiently strong radial field, or more accurately, for a field which satisfies the condition

$$\mu \left|\frac{K_l'(k_z a)}{K_l(k_z a)}\right| \frac{I_l'(k_z a)}{I_l(k_z a)} H_{1r}^2 > \mu_0 \frac{(\mathbf{H}_\tau \cdot \mathbf{k})^2}{k_z^2} + (G + \alpha k^2)\left[\mu \frac{I_l'}{I_l} + \mu_0 \left|\frac{K_l'}{K_l}\right|\right] \frac{\mu_0}{(\mu-\mu_0)^2 k_z},$$

the equilibrium cylindrical surface becomes unstable. To determine the limits of stability it is necessary to minimize the condition obtained with respect to all possible values of k_z and l, i.e., to determine the parameters of the most unstable perturbation and the corresponding critical value of the field H_{1r}^*, **determined by the condition**

$$(H_{1r}^*)^2 = \min_{k_z, l}\left(\left\{\mu_0 \frac{(\mathbf{H}_\tau \cdot \mathbf{k})^2}{k_z^2} + \left[\mu \frac{I_l'}{I_l} + \mu_0\left|\frac{K_l'}{K_l}\right|\right] \frac{\mu_0}{(\mu-\mu_0)^2 k_z} (G + \alpha k^2)\right\} \frac{1}{\mu} \frac{I_l}{I_l'} \left|\frac{K_l}{K_l'}\right|\right). \tag{2.4}$$

Here $\mathbf{k} = (0, l/a, k_z)$ and $\mathbf{H}_{1\tau} = \{0, H_{1\theta}, H_{1z}\}$.

Since condition (2.4) cannot generally be minimized analytically with respect to all k_z and l, we consider the limiting cases, enabling us to get rid of the Bessel functions.

Suppose $k_z a \gg 1$, but $l \sim 1$. In this case $I_l'/I_l \to 1$, $K_l'/K_l \to -1$, and (2.4) takes the form

$$\begin{aligned}(H_{1r}^*)^2 &= \min_{k_z, l}\left\{\frac{\mu_0}{\mu} \frac{(\mathbf{H}_\tau \cdot \mathbf{k})^2}{k_z^2} + (G + \alpha k^2) \frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2 k_z}\right\} = \\ &= \min_{k, \beta}\left\{\frac{\mu_0}{\mu} \frac{H_\tau^2 \cos^2(\gamma-\beta)}{\cos^2\beta} + \left(\frac{G}{k} + \alpha k\right) \frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2 \cos\beta}\right\},\end{aligned} \tag{2.5}$$

where β is the angle between the wave vector $\mathbf{k}$ and the z axis ($-\pi/2 < \beta < \pi/2$), and γ is the angle between the tangential component of the field $\mathbf{H}_\tau$ and the z axis. Minimization of (2.5) with respect to k leads to the condition

$$(H_{1r}^*)^2 = \min_{\beta}\left\{\frac{\mu_0}{\mu} \frac{H_\tau^2 \cos^2(\gamma-\beta)}{\cos^2\beta} + 2\sqrt{\alpha G} \frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2 \cos\beta}\right\}, \tag{2.6}$$

where the k^* corresponding to the minimum is

$$k^* = \sqrt{G/\alpha} \tag{2.7}$$

and does not depend on β. The expression in curly brackets in (2.6) consists of two terms, each of which has a clear physical meaning with respect to its effect on the direction of the perturbations which are most "dangerous" for stability. The first term is minimum for $\beta = \gamma + \pi/2$; i.e., the tangential magnetic field eliminates perturbations propagating at right angles to its direction. The second term is minimum for $\beta = 0$; i.e., the axial symmetry of the problem eliminates perturbations propagating along the axis. Actually, perturbations with $l = 0$ are the most "dangerous" for the stability of an ordinary fluid. The interaction of these two mechanisms leads to a certain new direction of most "dangerous" perturbations.

If $\mathbf{H}_\tau = 0$, the first term is zero and the minimum occurs for $l = 0$; in this case,

$$(H_{1r}^*)^2 = 2\sqrt{\alpha G}\frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2}. \tag{2.8}$$

If $\mathbf{H}_\tau \neq 0$, differentiation of (2.6) with respect to β leads to a trigonometric equation for β^*,

$$\frac{\mu_0}{\mu}H_\tau^2\sin\gamma\cos(\beta^*-\gamma)+\sin 2\beta^*\sqrt{\alpha G}\frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2}=0, \tag{2.9}$$

determining the direction of the most "dangerous" perturbations. This direction is determined by the magnitude and direction of the field $\mathbf{H}_\tau$ and by other characteristics of the problem.

We note that for $\beta - \gamma \neq \pi/2$, the critical value of the field H_{1r}^* is larger than that given by (2.8); i.e., the tangential field $\mathbf{H}_\tau$ not only changes the direction of the most "dangerous" perturbations, but also produces a further reserve of stability.

As a matter of fact, if $\gamma = \pi/2$, the two physically distinct directions — normal to the field and along the axis of the jet — coincide, and $\beta = 0$. Since $\gamma - \beta = \pi/2$, the critical field is again given by (2.8); i.e., the azimuthal field actually does not affect perturbations with $k_z \neq 0$. If $\gamma = 0$, we find from (2.9) that $\beta = 0$; now, however, the critical field intensity is increased by $\mu_0 H_\tau^2/\mu$ in comparison with the case of a radial field only.

If $k_z a \gg 1$ and $l \gg 1$, $I_l'/I_l \approx k/k_z$, and $K_l'/K_l \approx -k/k_z$, and from (2.4) we obtain

$$(H_{1r}^*)^2 = \min_\beta\left\{\frac{\mu_0}{\mu}H_\tau^2\cos^2(\gamma-\beta)+2\sqrt{\alpha G}\frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2}\right\}, \tag{2.10}$$

where k^* is again determined by (2.7), and the most "dangerous" perturbations are those propagating in the direction of the transverse field $\mathbf{H}_\tau$. Equation (2.10), derived in the approximation $k_z a \gg 1$ and $l \gg 1$, as should be expected, agrees with the corresponding expression derived for a plane surface, the only difference being that G plays the role of ρg.

Perturbations with $k_z = 0$ described by Eq. (2.1) lead to instability of the jet if

$$H_{1r}^2\mu - H_{1\theta}^2\mu_0 > \frac{\mu_0(\mu+\mu_0)}{(\mu-\mu_0)^2}\frac{a}{s}\left(G+\alpha\frac{s^2}{a^2}\right),$$

from which

$$(H_{1r}^*)^2 = \frac{\mu_0}{\mu}(\mathbf{H}_\tau\cdot\mathbf{k}_0)^2+2\sqrt{\alpha G}\frac{\mu_0(\mu+\mu_0)}{\mu(\mu-\mu_0)^2}, \tag{2.11}$$

where $\mathbf{k}_0 = \{0, 1, 0\}$ and $k^* = s^*/a = \sqrt{G/\alpha}$.

Since the perturbations are periodic in θ, s^* must be an integer. If $a\sqrt{G/\alpha}$ is not an integer, the critical number s^* corresponds to the number $[a\sqrt{G/\alpha}]$ or $[a\sqrt{G/\alpha}] + 1$ for which H_{1r}^* is smaller. Here $[a]$ denotes the integral part of a. We see that the presence of the field $H_{1\theta}$ increases the threshold values of the field above which perturbations with $k_z = 0$ are unstable.

Thus, in contrast with a plane surface, a tangential field has a significant effect on the appearance of instability of an axisymmetric jet. As a matter of fact, for a field $\mathbf{H}_\tau = \{0, H_\theta, H_z\}$, the direction of the most "dangerous" perturbations with $k_z \neq 0$ is not perpendicular to $\mathbf{H}_\tau$, as was the case for a plane surface, because of the effect of axial symmetry, and as a result the critical value of the field H_{1r}^* becomes larger than in the absence of a tangential field $\mathbf{H}_\tau$. If the component of the tangential field H_θ is different from zero, the instability threshold increases also for perturbations with $k_z = 0$. Then, whether the perturbation with $k_z = 0$ or the one with $k_z \neq 0$ begins to develop first depends on a comparison of corresponding critical values of the field, but in general both thresholds are higher than for $\mathbf{H}_\tau = 0$. Once more we note that the change in threshold occurs when the field $\mathbf{H}_\tau$

has both azimuthal and axial components. The presence of only one component (the threshold remains unchanged) changes the form of the developing perturbations; the most unstable perturbations are those with a wave vector perpendicular to the existing component of the tangential field. If there is no tangential field, the magnetic permeability of the fluid determines whether the perturbation with $k_z = 0$ or the one with $l = 0$ is more unstable in a radial field. A comparison of the minima of Eqs. (2.4) for $l = 0$ and (2.11), for example, for a jet of radius $a = 5\sqrt{\alpha/G}$, shows that for $\mu/\mu_0 < 1.35$ the perturbations with $k_z = 0$ are the more unstable, and for $\mu/\mu_0 > 1.35$ the opposite is the case: The threshold is lower for perturbations with $l = 0$.

The stabilizing effect of a uniform axial magnetic field was easily observed experimentally. In our experiments a vertical cylindrical jet of magnetizable fluid has a diameter of 1 mm at the critical point where it breaks up into drops 58 mm from the point of efflux and was completely stabilized over the whole experimental portion 150 mm long by a uniform vertical magnetic field of 130 kA/m. The magnetization of the fluid was 12 kA/m.

In addition, by changing the flow rate of the fluid, an initial stream of drops was transformed into a jet by turning on a magnetic field of the intensity indicated.

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